

Energy Computation in Wormhole Background with the Wheeler-DeWitt Operators

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We investigate the possibility of computing energy by means of operators associated to the *Wheeler-DeWitt* equation. By choosing three dimensional wormholes as a framework, we apply such calculation scheme to the black hole pair creation. We compare our results with the recent ones appeared in the literature.

1. Introduction

In recent years, a growing attention is devoted to the subject of wormholes in its various aspects. Here we would like to use such objects together with a variational calculation to probe the quantum gravitational vacuum. As an application, we will consider the pair creation problem for neutral black holes, whose rate is determined by the following formula

$$P \simeq |e^{-I_{cl}}|^2 \simeq |e^{-(\Delta E)(\Delta t)}| \quad (1)$$

The sourceless Einstein's equations, with and without cosmological constant, select two types of spherical symmetric solutions representing a hole in spacetime:

- the Schwarzschild solution (S)
- the Schwarzschild-deSitter solution (SdS).

According to [7], we shall call such solutions wormholes. Although the exclusion of matter fields is far from a realistic picture of quantum gravity, we use such a representation to study quantum effects in the very early universe. In static coordinates these solutions assume the form

$$ds^2 = \pm F(r) dt^2 + F^{-1}(r) dr^2 + r^2 d\Omega^2, \quad (2)$$

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where “+” is for the Euclidean signature while “−” is for the Lorentzian one. The function $F(r)$ is such that

$$\begin{cases} 1 - \frac{2MG}{r} & (S) \\ 1 - \frac{2MG}{r} - \frac{1}{2}\Lambda r^2 & (SdS) \end{cases}, \quad (3)$$

M is the parameter describing the mass of the wormhole, G is the gravitational constant, Λ is the positive cosmological constant and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. By observing that, the constant “time” section of (2) is represented by a three dimensional wormhole which maintains metric and topology for both signatures, we are led to consider 3D wormholes as preferred frame for probing the vacuum. Nevertheless, for our purposes, we shall make the choice of completing the spacetime in the compact Euclidean direction. More details on this subject can be found in Ref. [13]

2. Instantons and Wormholes

We report here the analogies between gravitational instantons and wormholes related to the metrics (topologies) under examination.

2.1. Instantons

The instantons associated to the problem under examination are

| | |
|--------------------------------------|--------------------------------------|
| a) <i>Nariai</i> (N) | $\mathbf{S}^2 \times \mathbf{S}^2$ |
| b) <i>deSitter</i> (dS) | \mathbf{S}^4 |
| c) <i>Gibbons – Hawking</i> (GS) | $\mathbf{R}^2 \times \mathbf{S}^2$ |
| d) <i>FlatSpace</i> (FS) | $\mathbf{R}^3 \times \mathbf{S}^1$. |

In particular in a) and in b) no boundary terms are needed because the corresponding topologies

are compact. Because the (SdS) metric tends asymptotically to the (dS) metric, we can consider the latter one as a reference background. Moreover, for regularity reasons we refer to the (N) instanton that is the extreme (SdS) instanton [6]. For these two instantons (topologies) the value of the action, for $\Lambda > 0$, is

$$\begin{aligned} -\frac{3\pi}{G\Lambda} & \quad (dS) \\ -\frac{2\pi}{G\Lambda} & \quad (N) \end{aligned} \quad (4)$$

In c) and in d) the action needs a boundary term, otherwise the path integral is meaningless. Here $\Lambda = 0$ and

$$I = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} [K], \quad (5)$$

where $[K]$ is the difference in the trace of the second fundamental form of $\partial\mathcal{M}$ in the metric g and the metric η referred to the flat space. Then $I = 4\pi GM^2$, where we have used the fact that the Euclidean “time” is periodic with period $8\pi GM$ and the fact that the hypersurface is bounded by the surface $r = r_0$.

2.2. Wormholes

Turning to the wormhole sector, the scalar curvature appearing in the action is better described in terms of *lapse* and *shift* variables. The contribution for these metrics comes only from a boundary term. This is a consequence of the fact that we have opened the hypersurfaces, i.e. the manifold is no more a compact object. Then, the relation between the four dimensional spacetime and the three dimensional space plus one compact time, with and without cosmological constant, is

$${}^{(4)}R = \frac{2}{N} \nabla^2 N \quad (6)$$

To check that the action contribution is the same of the Sec.(2.1), we have to integrate over the period the boundary term. This process gives the relation between the four compact dimension and the three plus one compact time dimensions.

a) (dS) vs. \mathbf{S}^4 topology.

The line element is of the form of the eq (1). By identifying $N^2(r) = F(r) = 1 - \frac{\Lambda}{3}r^2$, the action becomes

$$\begin{aligned} I &= -\frac{1}{8\pi G} \int d\tau \int d^3x \left[\partial_i \left(\sqrt{3} g g^{ij} \partial_j N \right) \right] \\ &= -\frac{\Lambda}{6G} \int d\tau r^3 \Big|_{\sqrt{\frac{3}{\Lambda}}} = -\frac{3\pi}{\Lambda G}. \end{aligned} \quad (7)$$

b) (N) vs. $\mathbf{S}^2 \times \mathbf{S}^2$ topology.

By identifying $N^2(r) = F(r) = 1 - \Lambda r^2$, we obtain

$$I = -\frac{1}{8\pi G} \left(\frac{2\pi}{\sqrt{\Lambda}} \right) \left(4\pi \frac{1}{\Lambda} \right) \int d\tau = -\frac{2\pi}{G\Lambda}. \quad (8)$$

c) (S) vs. $\mathbf{R}^2 \times \mathbf{S}^2$ topology.

By identifying $N^2(r) = F(r) = 1 - \frac{2M}{r}$, one obtains directly the same result of the instanton because in that case the only contribution comes from the boundary.

3. Calculation Scheme and Application to the Black Hole Pair Creation

From the previous section we have seen how to recover instanton results starting from 3D wormhole at the classical level. On these grounds we approach quantum gravitational effects:

- Our framework will be a variational calculation applied to gaussian wave functionals $\Psi \{g_{ij}(x)\}$. The central point is the evaluation of

$$\frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad (9)$$

where H is the total Hamiltonian which contains two classical constraints

$$\begin{cases} \mathcal{H} = 0 \\ \mathcal{H}_i = 0 \end{cases} \quad (10)$$

The operator version of the first of the two previous constraints is the *Wheeler-DeWitt* equation. It is immediate to see that the

calculation of the quantity expressed in eq.(9) is ill posed. However, we can proceed backward by evaluating the average of H in an enlarged *nonphysical* space, postponing the projection on the physical Hilbert space by imposing the *Wheeler-DeWitt* equation.

- We will consider small perturbations with respect to the constant “time” section (S) and (N) metric that we denote as \tilde{g}_{ij} . Then the perturbations will be in the *three dimensional* space and

$$g_{ij} = \tilde{g}_{ij} + h_{ij}, \quad (11)$$

with

$$\langle \Psi | g_{ij} | \Psi \rangle = \tilde{g}_{ij}. \quad (12)$$

- In this context, the *shift* function is zero, then only \mathcal{H} is non vanishing in the *enlarged space*. We will denote this operator \mathcal{H}_{WDW} . The expansion of \mathcal{H}_{WDW} to one loop gives rise to the second order differential operator acting only on *gravitons* (*TT sector*).
- The analysis of the discrete spectrum for

$$(-\Delta \delta_j^a + 2R_j^a) h_a^i = -E^2 h_j^i \quad (13)$$

gives one negative squared eigenvalue for both metrics, where R_j^a is the Ricci tensor in $3D$ and Δ is the Laplacian in curved space.

$$\begin{array}{cc} -.24 (GM)^{-2} & (GH) \\ -2\Lambda & (N). \end{array} \quad (14)$$

- Negative squared eigenvalues appear for the graviton sector in $4D$ by means of the expansion of the Euclidean action at one loop [4,3,9]

$$\begin{array}{cc} -.19 (GM)^{-2} & (GH) \\ -2\Lambda & (N). \end{array} \quad (15)$$

- Presence of negative modes in three dimensions plus one *compact* dimension is guaranteed by the same analysis in a Kaluza-Klein manifold with one dimension suppressed.[11]
- Negative modes *imply* the decay from the *false vacuum* to the (possible) *true vacuum*. The decay probability per unit time and unit volume is defined, at least semiclassically as

$$\Gamma = A \exp(-I_{cl}), \quad (16)$$

where A is the prefactor coming from the saddle point evaluation and I_{cl} is the classical part of the action.

- Neglecting the prefactor [10], the approximate value of the probability of nucleating a black hole in the different instantons (topologies) is

$$\Gamma \sim \begin{cases} \exp -4\pi M^2 G & (GH) \\ \exp -\frac{\pi}{\Lambda G} & (N) \end{cases} \quad (17)$$

In any case the “hot” space cannot describe the ground state, therefore a topology change comes into play and a black hole nucleation can be realized when we consider the *hot flat space*, while black hole pair creation is the mechanism related to the *hot de Sitter space*. It is immediate to recognize that eq. (17) represents the decay rate calculated with the no-boundary prescription of Hartle-Hawking. In fact, following Ref. [2], we define

$$\Gamma = \frac{P_{SdS}}{P_{deSitter}} = \exp -\frac{\pi}{G\Lambda}. \quad (18)$$

However in eq. (1), the form of the decay probability can be expressed in terms of the wavefunction solving the WDW equation. For this purpose we need to project out the non-physical states from the initial enlarged space. A possibility to do this is by means of the following choice [8]

$$\Psi[g_{ij}] = \int_{\gamma} dN e^{-N\omega} \Phi[g_{ij}]. \quad (19)$$

However, since we have introduced boundary terms the previous equation has to be modified with

$$\Psi[g_{ij}] = \exp(I_{b.t.}) \int_{\gamma} dN e^{-N\omega} \Phi[g_{ij}], \quad (20)$$

where

$$I_{b.t.} = \frac{1}{8\pi} \int_{\tau=0} d^3x \sqrt{^3g} K. \quad (21)$$

The only difference with the Hartle-Hawking wave function is that $\Phi[g_{ij}]$ is a trial wave functional of the gaussian type [5]. By repeating the same procedure of cutting half of the instanton, we recover the results that lead to eq. (18). The same approach can be applied to the (GH) instanton [12].

4. Conclusions and Outlooks

- Wormholes without matter fields could be significant in the very early universe
- The probability of decay is relevant when $\Lambda \sim 1$ (in Planck's units)
- Nucleation happens in the hot space.

We need to investigate:

- The cold space, i.e. $T = 0$.
- Higher order corrections
- The conformal factor
- Matter field contribution
- Multi-wormholes \longleftrightarrow spacetime foam.

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